PHYSICAL REVIEW VOLUME 135, NUMBER 5B 7 SEPTEMBER 1964

Time and the S Matrix*

DAVID BRANSON
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, England
(Received 12 March 1964)

A method is introduced for defining the time duration of a scattering process. Using this definition, the connection between causality and analyticity is discussed. An application to three-particle scattering leads to a discussion of the necessity for the existence of stable and unstable particle poles, and finally an analysis is given of rescattering processes and the generation of two-particle normal thresholds, with a view to obtaining Feynman's $i\epsilon$ prescription for deciding which is the physical boundary value.

1. INTRODUCTION

CONSIDERABLE progress has been made recently towards the establishment of a self-contained S-matrix theory, entirely independent of quantum field theory.[1] The reason for such work is the growing feeling among some physicists that field theory is not the most appropriate tool for studying elementary particles; some doubt the validity of field theory,[2] and while the doubt remains it seems worthwhile to try to remove the dependence that the S matrix has formerly had on field theory. Even if the objections to the latter were to be overcome and the two approaches found to be equivalent, S-matrix theory would still be very attractive as it deals directly with measurable quantities and is therefore that much nearer physics.

Space-time coordinates are rarely mentioned in S-matrix theory—the matrix elements are considered as functions of the energies and momenta of the particles involved in a scattering process. However, it is clear that space-time must enter the theory somewhere: For example, when calculating cross sections we find it necessary to interpret a four-dimensional δ function of zero argument as the product of the volume of space in which the experiment was performed and the time it took; we must incorporate into the theory the fact that the forces we are dealing with have a short range (this question has been considered by Stapp[3]; a detailed paper on the same topic by Wichmann and Crichton has recently appeared[4]); we may wish to discuss two or more scattering processes that take place successively; unstable particles have associated with them a life*time*; the analytic properties of matrix elements are supposed to be connected with causality which depends for its description on the possibility of some localization in space and time.

It appears that a preoccupation with momentum space has led to the necessity of postulating some rather artificial "axioms" in S-matrix theory. These axioms usually embody results that can be proved in finite-order perturbation theory, and they are presumably adopted on the principle that "whatever perturbation theory predicts is most likely to be true." A little investigation of field theory shows that these particular results depend on space-time properties. The object of this paper is to introduce time into S-matrix theory, and hence to find a more physical basis for some of its axioms, and so to reduce its dependence on field theory.

In Sec. 2, we describe the formalism for introducing an idealized "microscopic" time into the theory and show how it can be modified to describe a more realistic "macroscopic" time. We show how these ideas are consistent with the usual probability interpretation of the scattering amplitude and as a simple example we consider the time duration of a resonance scattering process.

Analyticity is one of the principal weapons of the S-matrix theory, and it is usually assumed that the justification for postulating analyticity is that it is somehow connected with causality. We investigate this assumption in Sec. 3, and show that in a nonrelativistic theory a simple intuitive causality condition implies that the scattering amplitude has a continuation into the upper half-energy plane. However, when we take account of relativity, the causality condition must be modified, and the analytic properties we deduce are either nonexistent or much restricted.

Section 4 is principally a collection of results obtained by other authors on the topic of the connectedness structure, which is the means whereby we can take account of macroscopic space in a plane-wave theory. This forms an introduction to Sec. 5, in which we consider three-particle scattering. It is known that unitarity predicts the existence of a physical region infinity in the three-particle scattering amplitude, and we show how the interpretation of two successive two-particle scattering events indicates that this infinity is, in fact, a pole. We also derive the usual relation between the lifetime of an unstable particle and the imaginary part of its mass.

Section 6 contains a detailed discussion of the two-particle branch point in the scattering amplitude. We first define a reduced amplitude in which this singularity

* The research reported in this document has been sponsored in part by the Air Force Office of Scientific Research, OAR, under Grant No. AF EOAR 63-79 with the European Office of Aerospace Research, U. S. Air Force.

[1] For a recent review see P. V. Landshoff, University of Cambridge (unpublished).
[2] G. F. Chew, Sci. Progr. (London) **51**, 529 (1963).
[3] H. P. Stapp, University of California (to be published).
[4] E. H. Wichman and J. H. Crichton, Phys. Rev. **132**, 2788 (1963).

is absent, and then show how the branch point is generated by rescattering processes; we derive the result, familiar in perturbation theory, that the threshold is depressed infinitesimally into the lower half-plane.

2. DESCRIPTION OF TIME

We describe the state of a system of particles in a particular Lorentz frame by the total energy E, and the remaining variables needed to determine the particle momenta, which we denote collectively by a, b, etc. Then, if the relative probabilities of the different outcomes of an initial experiment and a final experiment are given by $|A_i(E',a)|^2$, $|A_f(E,b)|^2$, respectively, the S matrix is defined by[5]

$$A_f(E,b)=\sum_{E',a} S(E,b;E',a)A_i(E',a). \quad (1)$$

This definition incorporates the superposition principle. Using the definition of the T matrix given by

$$S(E,b;E',a)=\delta(E-E')\delta(b-a) +i\delta(E-E')\delta^3(\mathbf{P}-\mathbf{P}')T_{ab}(E),$$

where $\mathbf{P}$, $\mathbf{P}'$ are the total three-momenta in the final and initial states, (1) becomes

$$A_f(E,b)=A_i(E,b)+i\textstyle\sum_a T_{ab}(E)A_i(E,a), \quad (2)$$

where we now take the sum over only those values of a which ensure $\mathbf{P}=\mathbf{P}'$.

In nonrelativistic quantum mechanics, when we consider the uncertainty principle or when we consider the Fourier transformation of variables, time and energy are conjugate variables, and Stapp[5] believes that in S-matrix theory also, time and energy are related by Fourier transformation. We accordingly define a variable t which possesses this expected property of time; we then introduce an extra physical assumption into the theory by requiring t to have some other properties of time. On the basis of this hypothesis we find we can then obtain consistent, sensible results and the theory now has sufficient strength to produce the results mentioned in the previous section.

We therefore formally define the function $\tilde{A}_i(t,a)$ by the transformation

$$\tilde{A}_i(t,a)=\int_{-\infty}^{\infty} dE e^{-iEt}A_i(E,a). \quad (3)$$

The choice of sign in the exponential is arbitrary. If we choose the opposite sign, the appropriate results are reversed. This is at least not discouraging, as if we change the sign of the time in field theory the corresponding results are also reversed.

If we apply the same transformation to Eq. (2) we obtain, using the convolution property of Fourier integrals,

$$\tilde{A}_f(t,b)=\tilde{A}_i(t,b)+i\int_{-\infty}^{\infty} dt' \sum_a \tilde{T}_{ab}(t-t')\tilde{A}_i(t',a), \quad (4)$$

where

$$\tilde{T}_{ab}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dE e^{-iE\tau}T_{ab}(E). \quad (5)$$

The inverse transformation is given by

$$T_{ab}(E)=\int_{-\infty}^{\infty} d\tau e^{iE\tau}\tilde{T}_{ab}(\tau). \quad (6)$$

It will be seen that Eq. (5) involves knowledge of $T_{ab}(E)$ for E below the physical threshold. We may at present define $T_{ab}(E)$ in this range arbitrarily, but we shall find, in the next section, that we will require the values that $T_{ab}(E)$ takes in the unphysical region to be an analytic continuation of those it takes in the physical region.

If t were connected with the time, Eq. (4) would describe the linear dependence of a final amplitude at time t on initial amplitudes at time t'. It is again encouraging that this relationship is independent of the origin from which t is measured; this is clearly a necessary property if t is, in fact, the time.

We therefore make the physical assumption that we can interpret t in Eq. (4) as the time. More precisely, we assume that the probability of a reaction taking place with the appropriate variables having values a, b, and such that the particles are interacting for a time between τ and $\tau+d\tau$ is proportional to

$$|\tilde{T}_{ab}(\tau)|^2 d\tau.$$

(Since the forces we are considering are short range, the particles interact for only a finite time.)

We have here defined essentially a microscopic time variable, which is somewhat unsatisfactory from the point of view of the uncertainty principle. In order to define precisely the time of interaction we have had to abandon all knowledge of the energy [this is why the integration in Eq. (5) must be taken to $-\infty$]. However, in any experiment, we always retain some knowledge of the energy, and there is then a finite uncertainty in both the energy and the time of interaction. Hence, it seems more realistic to modify Eq. (5) so as to define a macroscopic time. One method we could employ is to write

$$\tilde{T}_{ab}(\tau,E_0,\sigma)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dE e^{-iE\tau}e^{-(E-E_0)^2/\sigma^2}T_{ab}(E), \quad (7)$$

where we now could interpret

$$|\tilde{T}_{ab}(\tau,E_0,\sigma)|^2 d\tau$$

as being proportional to the probability of the given reaction taking place where the energy is approximately

[5] H. P. Stapp, Phys. Rev. **125**, 2139 (1962).

E_0 with an uncertainty of order σ (which depends on the experimental situation) and the time of interaction is approximately τ with a corresponding uncertainty.

One advantage of this definition is that the Gaussian factor now suppresses the values of $T_{ab}(E)$ for E below threshold in Eq. (7).

We now calculate the total probability of a reaction occurring with any value of τ for E distributed about E_0. This is proportional to

$$\int_{-\infty}^{\infty} |\tilde{T}_{ab}(\tau,E_0,\sigma)|^2 d\tau .$$

Using Eq. (7) this becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dE e^{-2(E-E_0)^2/\sigma^2}|T_{ab}(E)|^2 ,$$

which is just the result we would expect, and so this is further evidence in favor of our hypothesis being correct.

The most probable value for the time of interaction is given by

$$\langle\tau\rangle=\int_{-\infty}^{\infty} d\tau\tau|\tilde{T}_{ab}(\tau,E_0,\sigma)|^2\Big/\int_{-\infty}^{\infty} d\tau|\tilde{T}_{ab}(\tau,E_0,\sigma)|^2 .$$

When we again use Eq. (7) this reduces to

$$-(i/2)\int_{-\infty}^{\infty} dE[T_{ab}^*(E)T_{ab}'(E)-T_{ab}^{*\prime}(E)T_{ab}(E)]e^{-2(E-E_0)^2/\sigma^2}\Big/\int_{-\infty}^{\infty} dE T_{ab}^*(E)T_{ab}(E)e^{-2(E-E_0)^2/\sigma^2} .$$

If the Gaussian is sharply peaked, this is approximately equal to $\mathrm{Re}\{-iT'(E_0)/T(E_0)\}$ which is similar to the Wigner time-delay formula.[6]

As a crude application of the ideas developed in this section, we consider a scattering amplitude which has a resonance at $E=E_r$. Then the function

$$e^{-(E-E_r)^2/\sigma^2}T_{ab}(E)$$

changes relatively quickly in the neighborhood of E_r, so that when we make a Fourier analysis, the "high-frequency" components are larger than if the function had varied less violently. This implies that $\tilde{T}_{ab}(\tau,E_r,\sigma)$ is enhanced at large values of τ, corresponding to the fact that two particles resonating interact for a relatively long time.

3. ANALYTICITY

One of the "axioms" of S-matrix theory is that the matrix elements possess certain analytic properties. It is generally assumed (by analogy with field theory and with dispersion theory in optics) that this analyticity is connected with causality. We are now in a position to investigate this assumption more closely.

Consider first our definition of microscopic time given by Eqs. (3), (4), (5), and (6). Our ideas of causality suggest that a final state associated with time t can depend on an initial state associated with time t' only if $t>t'$. This implies, by Eq. (4)

$$\tilde{T}_{ab}(t-t')=0 \quad \text{for} \quad t<t' . \tag{8}$$

We could call this the "microscopic causality condition." If it holds then Eq. (6) becomes

$$T_{ab}(E)=\int_0^{\infty} d\tau e^{iE\tau}\tilde{T}_{ab}(\tau) .$$

If this integral converges for real E, then it certainly converges also for E in the upper half-plane, for then the integrand contains a factor $e^{-\gamma\tau}$ ($\gamma=\mathrm{Im}E>0$) which assists the convergence. It therefore seems we have shown that $T_{ab}(E)$ has an analytic continuation, regular in the upper half E plane. We see also that the causality condition implies that the values of $T_{ab}(E)$ for E below threshold in Eq. (5) must be the analytic continuation via the upper half-plane of the values it takes in the physical region.

Suppose, however, that we were working in the center-of-mass frame where $E=s^{1/2}$. Then we would have predicted the absence of a left-hand cut in the s plane. The left-hand cut arises from singularities associated with the crossed channels, and so the microscopic causality condition would lead to no contradiction if we were considering a theory in which crossing were absent. Since the crossing property arises from the relativistic nature of the theory, it seems that relativity must be one of the factors that invalidates the microscopic causality condition.

A combination of the uncertainty principle and special relativity predicts that energy fluctuations may be so large as to create short-lived "virtual" particles. We usually describe this situation by saying that each particle is surrounded by a "cloud" of virtual particles. It is therefore possible for two particles to interact through their "clouds," i.e., by the exchange of one or more virtual particles, and it is just these processes that give rise to the crossed-channel singularities. If the "clouds" (and therefore also the crossed-channel singularities) were absent, all interactions would be direct, and we could define the time of interaction precisely and use the microscopic causality condition, obtaining no contradictions. However, as soon as we consider the possibility of interaction through "clouds," the crossed-channel singularities appear, but clearly it is no longer

[6] E. P. Wigner, Phys. Rev. **98**, 145 (1955). See also M. L. Goldberger and K. M. Watson, *ibid.* **127**, 2284 (1962).

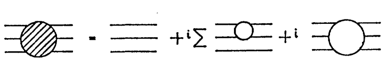


FIG. 1. Resolution of three-particle scattering.

meaningful to talk of a precise time of interaction: two "clouds" interact for a rather indefinite time, and hence it must be incorrect to suggest that $\tilde{T}_{ab}(\tau)$ has a sharp cutoff at $\tau=0$. We could remedy the situation by replacing Eq. (8) by a "macroscopic causality condition":

$$\tilde{T}_{ab}(\tau)\to 0 \quad \text{as} \quad \tau\to -\infty\,.$$

Then we can see from Eq. (6) that even if $\tilde{T}_{ab}(\tau)$ decreases for negative τ like τ^{-2N} for any positive N, we are unable to deduce any analyticity property for $T_{ab}(E)$. However, suppose $\tilde{T}_{ab}(\tau)$ decreases like $e^{\lambda\tau}(\lambda>0)$, then Eq. (6) implies that $T_{ab}(E)$ is analytic in the strip

$$0<\mathrm{Im}E<\lambda\,.$$

This would be the situation, for instance, if we were working in the center-of-mass frame for two-particle elastic scattering, keeping $\cos\theta$ constant in the range $-1<\cos\theta<1$. Then, provided the quantum numbers forbid single-particle poles, the "left-hand" singularities lie along the positive imaginary E axis, and provided the physical values of $T_{ab}(E)$ are given by passing above all the right-hand singularities (see Sec. 6) we do indeed have analyticity in a strip whose width is controlled by the nearness of the crossed-channel two-particle thresholds to the real E axis. As $\cos\theta$ approaches ± 1, one of these thresholds approaches nearer the real axis, i.e., λ is reduced, and so $\tilde{T}_{ab}(\tau)$ decreases less sharply for negative τ. This result is to be expected, as $\cos\theta\simeq\pm 1$ implies small momentum transfer in one of the crossed channels; if the exchanged particles carry less momentum, the uncertainty principle predicts that they can survive longer; hence, the "cloud radius" is larger and so we would expect to be able to define the time of interaction less precisely.

Similarly, if the quantum numbers allow a pole, this will be nearer to the real E axis than the normal thresholds, and so λ will again be reduced. [If $\cos\theta$ is sufficiently near ± 1, the pole will lie on the real E axis, and then we cannot say that $\tilde{T}_{ab}(\tau)$ is bounded by an exponential.] This is again expected, as it requires less energy to create a single virtual particle than to create two; the single particle therefore survives longer and the "cloud radius" is again increased.

4. CONNECTEDNESS

In S-matrix theory, the initial and final states are taken to be eigenstates of momentum (i.e., plane waves; in practice they must be wave packets with a very small momentum spread). The uncertainty principle implies, therefore, that the positions of the particles are completely undetermined, and hence it is meaningless to talk about their localization. However, because we have no knowledge of the positions of the particles, and because the forces have a short range, we can make the physical assertion that there is an overwhelming probability that the particles never approach near enough to scatter. This is the basis of the so-called connectedness postulate,[3,4] and it is the means whereby a measure of macroscopic space can be introduced into the theory. If there is no interaction, the momenta of the particles in the final state are the same as in the initial state, and so the corresponding part of the S matrix consists just of energy-momentum conserving delta functions. If we remove these delta functions, there remains the interesting part of the matrix element–that describing the interaction (the "connected" part). When the non-analytic delta functions have been removed, the assumption is usually made that the connected parts are analytic functions.

When we consider three-particle scattering, we must take into account, not only the case where there is no interaction, but also the possibility of two particles scattering while the third is unaffected. We can write this diagrammatically as in Fig. 1, where the sum is over the three states in which each particle in turn is not scattered.

If we insert this into the unitarity equation, we find the connectedness postulate is consistent with it, for those terms in the equation involving disconnected parts (and hence delta functions) cancel out when we apply two-particle unitarity.

The analysis that now follows is due to Olive.[7] If the total energy is below the four-particle threshold, the unitarity equation reduces to the one in Fig. 2, where

$$T_{ab}(1)=T_{ab};\quad T_{ab}(2)=T_{ba}{}^*\,.$$

The last term on the right-hand side contains a mass-shell delta function coming from the phase-space factor in the sum over states. It follows that at least one other term in Fig. 2 must contain an infinity at the same point to balance the equation, and the only possibility is that the terms on the left-hand side also have this infinity.

Without any loss of generality, we can therefore assume that the first term on the left-hand side in Fig. 2 contains a term represented by Fig. 3(a) together with terms regular at

$$p_1{}^0+p_2{}^0-p_6{}^0=[(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_6)^2+m^2]^{1/2}\,,$$

where m is the mass of the intermediate particle, and —①— is a factor containing the infinity. It follows from Hermitian analyticity that the second term on the left-hand side in Fig. 2 contains a term represented in Fig.

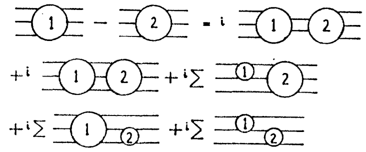


FIG. 2. Unitarity equation below four-particle threshold.

[7] D. I. Olive, Phys. Rev. 135, B745 (1964).

3(b) where

$$[-\textcircled{1}-]=[-\textcircled{2}-]^{*}. \tag{9}$$

We now pick out from Fig. 2 all the terms with this particular infinity (see Fig. 4).

Using two-particle unitarity, we obtain Fig. 5, and, on cancelling the scattering amplitudes,

$$i\text{———}=-\textcircled{1}- \;-\; -\textcircled{2}-. \tag{10}$$

But ——— is equal to $\delta^{(+)}((p_1+p_2-p_6)^2-m^2)$ where the $(+)$ indicates we take only the positive energy part of the delta function. We can therefore express the delta

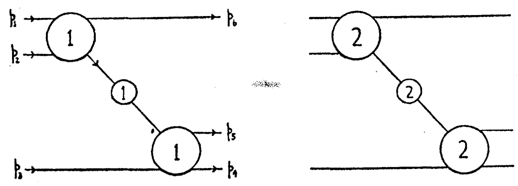


FIG. 3. Infinities present in three-particle scattering amplitudes.

function as

$$\lim_{\epsilon\to 0}\frac{1}{4\pi i[(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_6)^2+m^2]^{1/2}}\times\left\{\frac{1}{p_1{}^0+p_2{}^0-p_6{}^0-[(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_6)^2+m^2]^{1/2}-i\epsilon}-\frac{1}{p_1{}^0+p_2{}^0-p_6{}^0-[(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_6)^2+m^2]^{1/2}+i\epsilon}\right\},$$

where it is understood that we take the positive sign of the square roots, and $\epsilon>0$.

Hence Eqs. (9) and (10) have the solution

$$D^{(1)}=\lim_{\epsilon\to 0}\frac{1}{4\pi[(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_6)^2+m^2]^{1/2}}\times\left\{\frac{c}{p_1{}^0+p_2{}^0-p_6{}^0-[(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_6)^2+m^2]^{1/2}-i\epsilon}-\frac{1-c}{p_1{}^0+p_2{}^0-p_6{}^0-[(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_6)^2+m^2]^{1/2}+i\epsilon}\right\}, \tag{11}$$

where we have put

$$-\textcircled{1}-=D^{(1)}$$

and c is any real constant.

It is customary to choose arbitrarily $c=0$ (as Olive does[7]), so that $D^{(1)}$ is a pole. Clearly, if c is not equal to 0 or 1, $D^{(1)}$ is not analytic, and so we must modify our connectedness equation by subtracting the nonanalytic term from the connected part of the amplitude before we obtain an analytic function.

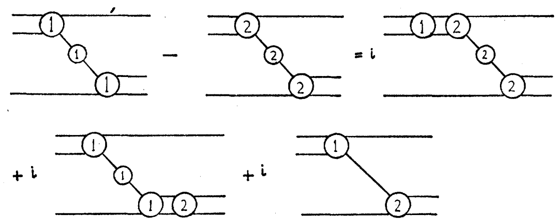


FIG. 4. Certain terms of the unitarity equation.

To decide what is the correct value of c we need to introduce some extra physics. We assume that the physical region infinity in the connected part of the amplitude represented by Fig. 3(a) must have a physical explanation, and the most obvious one is that the term represents a situation in which two of the incident particles scatter, and subsequently one of the emerging particles collides with the third incident particle.

If this is the correct interpretation, it is a very satisfactory one for, in general, a three-particle → three-particle scattering process is very unlikely, but if

$$p_1{}^0+p_2{}^0-p_6{}^0=[(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_6)^2+m]^{1/2},$$

then the infinity in the matrix element corresponds to the overwhelming dominance of processes consisting of

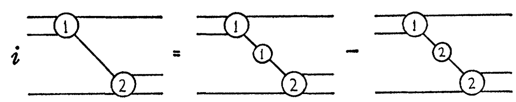


FIG. 5. Simplification of the terms selected.

two two-particle scatterings occurring successively. As Stapp[8] has emphasized, this property is essential to the interpretation of the theory. For example, a two-particle scattering amplitude is always determined experimentally by allowing at least one of the emerging particles to interact with a measuring apparatus. The whole process may thus be represented by Fig. 6. We are therefore looking at the three-particle connected amplitude just at the point where it is enhanced by the infinity, and the amplitude we are measuring is just one of the factors multiplying the infinite factor.

5. STABLE AND UNSTABLE PARTICLE POLES

In this section we apply the ideas of Sec. 2 to decide the value of the constant c in Eq. (11) assuming that the

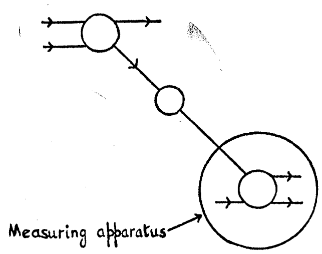


FIG. 6. Method of measuring a two-particle scattering amplitude.

[8] H. P. Stapp, University of California (to be published).

three-particle scattering amplitude contains an infinite term representing two successive scattering processes as in Fig. 3(a). For convenience we work in the center-of-mass frame of particles 1 and 2, and we shall vary the total energy E, while keeping p_3 and p_6 fixed. We assume all masses have the value m.

In the previous section we found that near

$$E=[\mathbf{p}_3{}^2+m^2]^{1/2}+2[\mathbf{p}_6{}^2+m^2]^{1/2}=\mathcal{E}(\text{say}),$$

the matrix element has dominant behavior

$$A(E)D^{(1)}(E)B(E),$$

where $A(\mathcal{E})$, $B(\mathcal{E})$ are two-particle scattering amplitudes (for simplicity, the dependence on variables other than E is not written explicitly).

According to Eq. (5), the probability that the whole process depicted in Fig. 3(a) has an interaction time τ depends on

$$\tilde{T}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dE e^{-iE\tau}A(E)D^{(1)}(E)B(E). \quad (12)$$

Using the convolution property, this becomes

$$\tilde{T}(\tau)=\int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} d\tau'' \tilde{B}(\tau-\tau')\tilde{D}^{(1)}(\tau'-\tau'')\tilde{A}(\tau''), \quad (13)$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{D}^{(1)}$ are defined in a similar way to $\tilde{T}$.

Since $\tilde{A}(\tau'')$, $\tilde{B}(\tau-\tau')$ are connected with the probabilities that the two-particle scattering interaction times are τ'', $\tau-\tau'$ respectively, and since τ is the over-all interaction time, Eq. (13) suggests that $\tilde{D}^{(1)}(\tau'-\tau'')$ is similarly connected with the probability that the time of flight of the intermediate particle is $(\tau'-\tau'')$. But this time of flight must be positive, and so we conclude

$$\tilde{D}^{(1)}(\tau)=0 \quad \text{if} \quad \tau<0.$$

From Eq. (5) and an application of Jordan's lemma, we see that this condition is satisfied provided

(a) $D^{(1)}(E)\to 0$ uniformly as $|E|\to\infty$, for E in the upper half-plane.

(b) $D^{(1)}(E)$ has no singularities in the upper half-plane. We can rewrite Eq. (11) as

$$D^{(1)}(E)=\lim_{\epsilon\to 0}\frac{1}{4\pi[\mathbf{p}_6{}^2+m^2]^{1/2}}\left\{\frac{c}{E-\mathcal{E}-i\epsilon}-\frac{1-c}{E-\mathcal{E}+i\epsilon}\right\},$$

and so (a) is certainly satisfied, and (b) is also, provided $c=0$.

We conclude that the infinity at $E=\mathcal{E}$ in the three-particle scattering amplitude is indeed a pole, displaced infinitesimally into the lower half-plane.

The fact that we do have a pole and not a delta function indicates that we notice the presence of the term represented in Fig. 3(a) when we are near to, but not exactly at, $E=\mathcal{E}$, and so it seems that Fig. 3(a) makes a contribution to the amplitude when the intermediate particle is not on the mass shell. The explanation is furnished by the uncertainty principle: since the intermediate particle does not exist for an infinite time, we cannot know its energy precisely. A related difficulty is that (12) involves values of the two-particle scattering amplitudes when one of the external particles is off the mass shell. Since S-matrix theory provides no unique continuation off the mass shell, this is a little unsatisfactory[9]; however, our final result is independent of the particular continuation we choose.

Having chosen $c=0$, we can evaluate $\tilde{D}^{(1)}(\tau)$ explicitly. If we now work in the center-of-mass frame of the intermediate particle we find

$$\tilde{D}^{(1)}(\tau)=i\theta(\tau)e^{-i(\omega_3+\omega_6+m)\tau}/4\pi m, \quad (14)$$

where ω_3, ω_6 are the energies of particles 3 and 6, respectively. Hence,

$$|\tilde{D}^{(1)}(\tau)|^2=\theta(\tau)/16\pi^2m^2,$$

showing that all (positive) times of flight are equally probable, as we would expect for a stable particle.

If, however, the particle were unstable, with lifetime $(2\Gamma)^{-1}$, we would expect to have

$$|\tilde{D}^{(1)}(\tau)|^2\propto\theta(\tau)e^{-2\Gamma\tau},$$

and we see from (14) that we would achieve this if the mass of the particle had an imaginary part $(-\Gamma)$. This result agrees with the usual practice of representing unstable particles by poles in the lower half-plane, the imaginary part of the pole position being related to the lifetime.

If we were more realistic and used Eq. (7) instead of Eq. (5), our results would alter quantitatively but not qualitatively.

Equation (13) becomes

$$\tilde{T}(\tau,E_0,\sigma)=\int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} d\tau'' \tilde{B}(\tau-\tau', E_0, \sigma_1) \times\tilde{D}^{(1)}(\tau'-\tau'', E_0, \sigma_2)\cdot\tilde{A}(\tau'',E_0,\sigma_3),$$

where

$$1/\sigma^2=1/\sigma_1{}^2+1/\sigma_2{}^2+1/\sigma_3{}^2.$$

Now $\tilde{D}^{(1)}(\tau,E_0,\sigma_2)$ does not vanish for negative τ; this is expected, because if we know τ only to a certain accuracy, negative values will be allowed. But if σ_2 is large, we expect the numerical value of $\tilde{D}^{(1)}$ to be little changed, and in fact, we see from the equation

$$D^{(1)}(E)e^{-(E-E_0)^2/\sigma_2{}^2}=\int_{-\infty}^{\infty} d\tau e^{iE\tau}\tilde{D}^{(1)}(\tau,E_0,\sigma_2),$$

[9] A fuller discussion of virtual particles and off-mass-shell amplitudes in S-matrix theory together with other topics related to this paper will be given in a forthcoming paper by R. J. Eden and P. V. Landshoff.

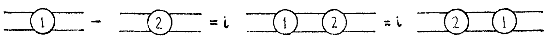


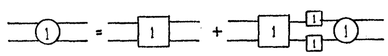
FIG. 7. Unitarity equation below three-particle threshold.

that if the integral is to converge, $\tilde{D}^{(1)}(\tau,E_0,\sigma_2)$ must die away for negative τ faster than any exponential, since the left-hand side is finite for E in the finite upper half-plane.

6. THE TWO-PARTICLE BRANCH POINT

We know that the scattering amplitude has branch points along the positive real energy axis, and it is necessary to know which side of the branch points we should pass in order to remain in the physical region. If we could prove that the physical value of the amplitude had a continuation into the upper half-plane, then

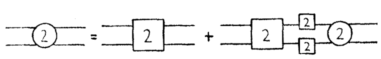
FIG. 8. Integral equation for reduced amplitude.



clearly the prescription would be that we must pass above all the branch points. However, we saw in Sec. 3 that we are, in general, unable to prove such an analytic property, and so in this section we tackle the problem in a different way by using unitarity.

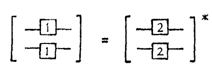
Below the three-particle threshold the unitarity equation may be written as in Fig. 7. In order to study the two-particle branch point in the physical amplitude we

FIG. 9. Conjugate integral equation.



cannot consider the terms on the right-hand side just as they stand, because they each represent the difference of two amplitudes, and because the branch point is contained in the "bubbles" as well as in the phase-space integration. We overcome the difficulties by defining (if possible) a reduced amplitude which does not itself contain the two-particle singularity. This amplitude is defined by the integral equation in Fig. 8 (assumed soluble), where in the last term we do not take the usual phase space integration, but another form which is to be determined.

FIG. 10. Condition for validity of Fig. 9.



The equation of Fig. 9, subject to Fig. 10, follows from Hermitian analyticity.

If the reduced amplitude does not possess the two-particle cut, between the two- and three-particle thresholds we have the equation in Fig. 11. Using Figs. 7 and

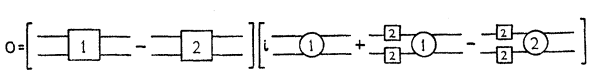


FIG. 11. Equation for amplitudes without two-particle cut.

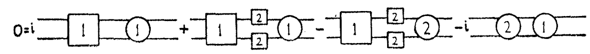


FIG. 12. Simplified equation.

9 this becomes Fig. 12. Using Figs. 7 and 8 we find the last term is equal to the expression in Fig. 13, so that the complete equation is given by Fig. 14. Since this equation is true for arbitrary values of the external momenta, we may conclude that the reduced amplitude does not have the two-particle cut provided the relation of Fig. 15 holds.

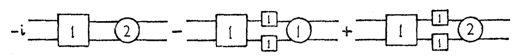


FIG. 13. Last term.

If we now formally iterate Fig. 8 (assuming the iteration converges) we obtain Fig. 16.

Since the reduced amplitude does not have the two-particle cut, we interpret it to be what the amplitude would be if there were no possibility of rescattering, and

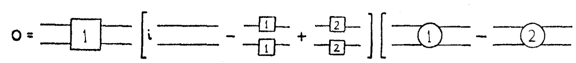
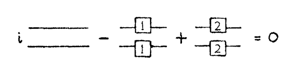


FIG. 14. Complete equation.

we now see that the two-particle singularity in the complete amplitude arises from processes in which rescattering occurs.

To study explicitly how the singularity is generated we consider the process represented in Fig. 17. For convenience we will work in the center-of-mass system, with total energy E. Let the two internal particles of mo-

FIG. 15. Condition for absence of two-particle cut.



menta q_1, q_2 have masses m_1, m_2 respectively. Then we consider first the expression

$$\int_{-\infty}^{\infty} d^4q_1 \int_{-\infty}^{\infty} d^4q_2 A(E)\delta(E-q_1{}^0-q_2{}^0)\delta^3(\mathbf{q}_1+\mathbf{q}_2)$$

$$\times\delta^{(+)}(q_1{}^2-m_1{}^2)\delta^{(+)}(q_2{}^2-m_2{}^2)B(E)=\int_{-\infty}^{\infty} d\mathbf{q}_1 A(E)$$

$$\times\frac{\delta(E-[\mathbf{q}_1{}^2+m_1{}^2]^{1/2}-[\mathbf{q}_1{}^2+m_2{}^2]^{1/2})}{4[\mathbf{q}_1{}^2+m_1{}^2]^{1/2}[\mathbf{q}_1{}^2+m_2{}^2]^{1/2}}B(E),$$

where now A, B represent reduced amplitudes.

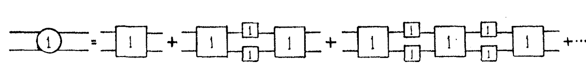


FIG. 16. Iterated integral equation.

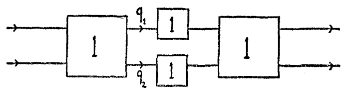


FIG. 17. A two-particle scattering process in which rescattering occurs once.

Putting

$$d\mathbf{q}=q^2dqd\Omega\,,$$

we obtain

$$\lim_{\epsilon\to 0}\frac{1}{8\pi i}\int d\Omega\int_0^\infty \frac{q^2dqA(E)}{[q^2+m_1^2]^{1/2}[q^2+m_2^2]^{1/2}} \times\left\{\frac{1}{E-[q^2+m_1^2]^{1/2}-[q^2+m_2^2]^{1/2}-i\epsilon} -\frac{1}{E-[q^2+m_1^2]^{1/2}-[q^2+m_2^2]^{1/2}+i\epsilon}\right\}B(E).$$

Therefore, as in Sec. 4, Figs. 10 and 15 imply that the contribution from Fig. 17 can be written as

$$\int d\Omega\int_0^\infty q^2dqA(E)D^{(2)}(E)B(E)\,,$$

where

$$D^{(2)}(E)=\lim_{\epsilon\to 0}\frac{1}{8\pi}\frac{1}{[q^2+m_1^2]^{1/2}[q^2+m_2^2]^{1/2}} \times\left\{\frac{c}{E-[q^2+m_1^2]^{1/2}-[q^2+m_2^2]^{1/2}-i\epsilon} -\frac{1-c}{E-[q^2+m_1^2]^{1/2}-[q^2+m_2^2]^{1/2}+i\epsilon}\right\},$$

for any real c.

To decide the value of c, we again appeal to the time interpretation, and we find exactly as in the previous section that if the intermediate particles are to have a positive time of flight we require $c=0$.

Therefore, Fig. 16 tells us that the two-particle scattering amplitude contains a term

$$\lim_{\epsilon\to 0}\frac{-1}{8\pi}\int d\Omega\int_0^\infty \frac{q^2dqA(E)}{[q^2+m_1^2]^{1/2}[q^2+m_2^2]^{1/2}} \times\left\{\frac{1}{E-[q^2+m_1^2]^{1/2}-[q^2+m_2^2]^{1/2}+i\epsilon}\right\}B(E).$$

We now investigate how this term produces the two particle singularity. Since we know $A(E)$, $B(E)$ are not singular at the two-particle threshold, we can treat them as constants for the purposes of the present discussion, and hence just confine our attention to the expression

$$\int_0^\infty \frac{q^2dq}{[q^2+m_1^2]^{1/2}[q^2+m_2^2]^{1/2}} \times\frac{1}{E-[q^2+m_1^2]^{1/2}-[q^2+m_2^2]^{1/2}+i\epsilon}\,. \qquad (15)$$

In the q plane there are square-root branch points at $\pm im_1$, $\pm im_2$ and poles at

$$q=\pm\{[(E+i\epsilon)^2-(m_1+m_2)^2] \times[(E+i\epsilon)^2-(m_1-m_2)^2]\}^{1/2}/2(E+i\epsilon)\,.$$

So long as E is greater than m_1+m_2 these poles neither pinch the contour of integration, nor coincide with an end point, and so function (15) has no singularity. However, when $E=m_1+m_2-i\epsilon$ the poles coincide with each other and with the end-point $q=0$ thus giving a singularity in function (15) and hence also in the term represented in Fig. 17. If E is taken twice around $m_1+m_2-i\epsilon$, function (15) returns to its original value, showing that the branch point is two-sheeted.

We may notice in passing that if E is reduced below m_1+m_2 through real values, the poles move along the imaginary q axis and disappear through branch cuts so that when the poles coincide at $E=m_1-m_2-i\epsilon$ the contour is not pinched, and there is no singularity. However, if E passes *underneath* $m_1+m_2-i\epsilon$, one of the poles drags the contour with it through a branch cut[10] and then the two poles *do* pinch it again when $E=m_1-m_2-i\epsilon$. This demonstrates how function (15) generates the well-known pseudothreshold reached by continuing underneath the two-particle branch point.

We may similarly analyze the other terms in Fig. 16 containing two-particle intermediate states and we find they all give singularities at $E=m_1+m_2-i\epsilon$. We conclude therefore that the two-particle normal threshold in the scattering amplitude occurs at $E=m_1+m_2-i\epsilon$, i.e., if the branch cut is drawn along the positive real axis, the physical values of the amplitude are obtained by approaching the branch cut from above.

It is possible to treat the problem of which way we should continue around the higher normal thresholds in a similar fashion. However, it is not so simple to define the reduced amplitude in this case, because of the presence of subenergy variables. A full discussion will be given in a later paper.

ACKNOWLEDGMENTS

I am very grateful to Dr. P. V. Landshoff for many helpful discussions. I should also like to thank Clare College for the award of a Research Studentship.

[10] It can be shown that the pinch which occurs in this case between the pole and the branch point does not give rise to a singularity.